

# Gorenstein Global Dimensions and Cotorsion Dimension of Rings

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**Abstract.** In this paper, we establish, as a generalization of a result on the classical homological dimensions of commutative rings, an upper bound on the Gorenstein global dimension of commutative rings using the global cotorsion dimension of rings. We use this result to compute the Gorenstein global dimension of some particular cases of trivial extensions of rings and of group rings.

**Key Words.** Gorenstein dimensions of modules; Gorenstein global dimensions of rings; cotorsion dimension of modules and rings;  $n$ -perfect rings.

## 1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unitary.

For a ring  $R$  and an  $R$ -module  $M$ , we use  $\text{pd}_R(M)$ ,  $\text{id}_R(M)$ , and  $\text{fd}_R(M)$  to denote, respectively, the classical projective, injective and flat dimensions of  $M$ . By  $\text{gldim}(R)$  and  $\text{wdim}(R)$  we denote, respectively, the classical global and weak global dimensions of  $R$ .

The Gorenstein homological dimensions theory originated in the works of Auslander and Bridger [1] and [2], where they introduced the G-dimension,  $\text{G-dim}_R(M)$ , of any finitely generated module  $M$  and over any Noetherian ring  $R$ . The G-dimension is analogous to the classical projective dimension and shares some of its principal properties (see [9] for more details). However, to complete the analogy an extension of the G-dimension to non-necessarily finitely generated modules is needed. This is done in [13, 14], where the Gorenstein projective dimension was defined over arbitrary rings (as an extension of the G-dimension to modules that are not necessarily finitely generated), and the Gorenstein injective dimension was defined as a dual notion of the Gorenstein projective dimension. And also to complete the analogy with the classical homological dimensions theory, the Gorenstein flat dimension was introduced in [16]. Since then, several results on the classical homological dimensions were extended to the Gorenstein homological dimensions. Namely, the majority of

works on the Gorenstein homological dimensions attempt to confirm the following meta-theorem (please see Holm's thesis [22, page v]): “*Every result in classical homological algebra has a counter part in Gorenstein homological algebra.*” (for more details see also [9, 10, 15, 21]). In line with this, the Gorenstein global dimensions of commutative rings were investigated in [4] (and [5]). It is proved, for any ring  $R$  [4, Theorems 3.1.3 and 3.2.1]:

$$\sup\{\text{Gfd}_R(M) \mid M \text{ } R\text{-module}\} \leq \sup\{\text{Gpd}_R(M) \mid M \text{ } R\text{-module}\} = \sup\{\text{Gid}_R(M) \mid M \text{ } R\text{-module}\}.$$

So, according to the terminology of the classical theory of homological dimensions of rings, the common value of  $\sup\{\text{Gpd}_R(M) \mid M \text{ } R\text{-module}\}$  and  $\sup\{\text{Gid}_R(M) \mid M \text{ } R\text{-module}\}$  is called *Gorenstein global dimension* of  $R$ , and denoted by  $\text{G-gldim}(R)$ , and the homological invariant  $\sup\{\text{Gfd}_R(M) \mid M \text{ } R\text{-module}\}$  is called *Gorenstein weak global dimension* of  $R$ , and denoted by  $\text{G-wdim}(R)$ .

The Gorenstein weak global and global dimensions are refinements of the classical weak and global dimensions of rings, respectively; that is [4, Propositions 3.11 and 4.5]:  $\text{G-gldim}(R) \leq \text{gldim}(R)$  and  $\text{G-wdim}(R) \leq \text{wdim}(R)$ , with each of the two inequalities becomes equality if  $\text{wdim}(R)$  is finite.

If  $R$  is a Noetherian ring, then [4, Corollary 2.3]:  $\text{G-wdim}(R) = \text{G-gldim}(R)$ , such that:

$$\text{G-gldim}(R) \leq n \iff R \text{ is } n\text{-Gorenstein}.$$

Recall that a ring  $R$  is said to be  $n$ -Gorenstein, for a positive integer  $n$ , if it is Noetherian with self-injective dimension less or equal than  $n$  (i.e.,  $\text{id}_R(R) \leq n$ ); and  $R$  is said to be Iwanaga-Gorenstein, if it is  $n$ -Gorenstein for some positive integer  $n$  (please see [15, Section 9.1]). Notice that 0-Gorenstein rings are the well-known quasi-Frobenius rings.

For a coherent ring  $R$ , we have [4, Theorem 4.11]:

$$\text{G-wdim}(R) \leq n \iff R \text{ is } n\text{-FC}.$$

Recall that a ring  $R$  is said to be  $n$ -FC, for a positive integer  $n$ , if it is coherent and  $\text{FP-id}_R(R) \leq n$  [8]; where  $\text{FP-id}_R(M)$  denotes, for an  $R$ -module  $M$ , the FP-injective dimension, which is defined to be the least positive integer  $n$  for which  $\text{Ext}_R^{n+1}(P, M) = 0$  for all finitely presented  $R$ -modules  $P$ . Notice that the 0-FC rings coincide (in commutative setting) with the IF-rings; i.e., rings over which every injective module is flat (please see [11, 23, 25]).

In this paper, we continue the study of the Gorenstein global dimensions of commutative rings started in [4] and [5]. The paper extends some results on the classical global homological dimensions to the Gorenstein global dimensions. To see that recall the following:

In [12], Ding and Mao introduced the cotorsion dimension of modules and rings, which are defined as follows:

**Definition 1.1 ([12])** *Let  $R$  be a ring.*

*The cotorsion dimension of an  $R$ -module  $M$ , denoted by  $\text{cd}_R(M)$ , is the least positive integer  $n$  for which  $\text{Ext}_R^{n+1}(F, M) = 0$  for all flat  $R$ -modules  $F$ .*

*The global cotorsion dimension of  $R$ , denoted by  $\text{cot.D}(R)$ , is defined as the supremum of the cotorsion dimensions of  $R$ -modules.*

The global cotorsion dimension of rings measures how far away a ring is from being perfect: the perfect rings are those rings over which every flat module is projective (please see [3]). Namely, we have, for a ring  $R$  and a positive integer  $n$ ,  $\text{cot.D}(R) \leq n$  if and only if every flat  $R$ -module  $F$

has projective dimension less or equal than  $n$  [12, Theorem 7.2.5 (1)]. In [17], a ring that satisfies the last condition is called  $n$ -perfect. So, we have:  $\text{cot.D}(R) \leq n$  if and only if  $R$  is  $n$ -perfect. Particularly,  $\text{cot.D}(R) = 0$  if and only if  $R$  is 0-perfect if and only if  $R$  is perfect.

The global cotorsion dimension of rings is also used to give an upper bound on the global dimension of rings as follows [12, Theorem 7.2.11]: For any ring  $R$ , we have the inequality:

$$\text{gldim}(R) \leq \text{wdim}(R) + \text{cot.D}(R).$$

The main result of this paper (Theorem 2.1) extends this inequality to the Gorenstein global dimensions of coherent rings. This result enables us to compute the Gorenstein global dimension of a particular case of trivial extensions of rings (Proposition 2.5).

In the end of the paper, we investigate the global cotorsion dimension of group rings (Theorem 2.9). This is used with the main result to compute the Gorenstein global dimension of a particular case of group rings (Proposition 2.11).

## 2 Main results

Our main result is the following:

**Theorem 2.1** *If  $R$  is a coherent ring, then:*

$$\text{cot.D}(R) \leq \text{G-gldim}(R) \leq \text{G-wdim}(R) + \text{cot.D}(R).$$

*In particular:*

- *If  $\text{cot.D}(R) = 0$  (i.e.,  $R$  is perfect), then  $\text{G-wdim}(R) = \text{gldim}(R)$ .*
- *If  $\text{G-wdim}(R) = 0$  (i.e.,  $R$  is an IF-ring), then  $\text{cot.D}(R) = \text{G-gldim}(R)$ .*

To prove this theorem, we need the following result, which is a generalization of the characterization of the Gorenstein projective dimension over Iwanaga-Gorenstein rings [18, Theorem 2.1].

**Lemma 2.2** *Let  $R$  be both an  $n$ -FC ring and an  $m$ -perfect ring, where  $n$  and  $m$  are positive integers. For any  $R$ -module  $M$ , we have the following equivalence, for a positive integer  $k$ :*  
 $\text{Gpd}_R(M) \leq k \Leftrightarrow \text{Ext}_R^j(M, P) = 0$  for all  $j \geq k + 1$  and all modules  $P$  with finite  $\text{pd}_R(P)$ .

In the proof of this lemma we use the notion of a flat preenvelope of modules which is defined as follows:

**Definition 2.3 ([15])** *Let  $R$  be a ring and let  $F$  be a flat  $R$ -module. For an  $R$ -module  $M$ , an homomorphism  $\varphi : M \rightarrow F$  is called a flat preenvelope, if for any homomorphism  $\varphi' : M \rightarrow F'$  with  $F'$  is a flat module, there is an homomorphism  $f : F \rightarrow F'$  such that  $\varphi' = f\varphi$ .*

The coherent rings can be characterized by the notion of a flat preenvelope of modules as follows:

**Lemma 2.4 ([15], Proposition 6.5.1)** *A ring  $R$  is coherent if and only if every  $R$ -module has a flat preenvelope.*

**Proof of Lemma 2.2.** The direct implication holds over arbitrary rings by [21, Theorem 2.20].

Conversely, consider an exact sequence of  $R$ -modules:

$$0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

where each  $P_i$  is projective. We have  $\text{Ext}^{n+i}(M, Q) \cong \text{Ext}^i(K_n, Q)$  for all  $i \geq 1$  and all modules  $Q$ . Then, to prove this implication, it is sufficient to prove it for  $k = 0$ . Then, we assume that  $\text{Ext}^i(M, P) = 0$  for all  $i \geq 1$  and all  $R$ -modules  $P$  with finite projective dimension, and we prove that  $M$  is Gorenstein projective. This is equivalent to prove, from [21, Proposition 2.3], that there exists an exact sequence of  $R$ -modules:

$$\alpha = 0 \rightarrow M \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots,$$

where each  $P^i$  is projective, such that  $\text{Hom}_R(-, P)$  leaves the sequence  $\alpha$  exact whenever  $P$  is a projective  $R$ -module.

The proof of this implication is analogous to the one of [18, Theorem 2.1 (1  $\Rightarrow$  4)]. For completeness, we give a proof here.

As usual (see for instance the proofs of [9, Theorems 4.2.6 and 5.1.7]), to construct the sequence  $\alpha$ , it is sufficient to prove the existence of a short exact sequence of  $R$ -modules:

$$0 \rightarrow M \rightarrow P^0 \rightarrow G^0 \rightarrow 0,$$

where  $P^0$  is projective, such that  $\text{Ext}^i(G^0, P) = 0$  for all  $i > 0$  and all  $R$ -modules  $P$  with finite projective dimension (and then the sequence  $\alpha$  is recursively constructed).

First, we prove that  $M$  can be embedded into a flat  $R$ -module. For that, pick a short exact sequence of  $R$ -modules  $0 \rightarrow M \rightarrow I \rightarrow E \rightarrow 0$ , where  $I$  is injective. For this  $I$  pick a short exact sequence of  $R$ -modules  $0 \rightarrow Q \rightarrow P \rightarrow I \rightarrow 0$ , where  $P$  is projective. Consider the following pullback diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & Q & = & Q & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & D & \rightarrow & P & \rightarrow & E \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & M & \rightarrow & I & \rightarrow & E \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Since  $I$  is injective,  $\text{pd}(I) < \infty$  (From [4, Theorem 4.11] and since  $R$  is  $m$ -perfect). Then,  $\text{pd}(Q) < \infty$ . By hypothesis,  $\text{Ext}(M, Q) = 0$ , and then the first vertical exact sequence is split, so  $M$  embeds into  $D$  which is an  $R$ -submodule of the projective (then flat)  $R$ -module  $P$ .

The fact that  $M$  embeds into a flat  $R$ -module implies, from Lemma 2.4 and Definition 2.3, that  $M$  admits an injective flat preenvelope  $\varphi : M \rightarrow F$ . For such flat  $R$ -module  $F$ , consider a short exact sequence of  $R$ -modules  $0 \rightarrow H \rightarrow P^0 \xrightarrow{f} F \rightarrow 0$ , where  $P^0$  is projective, then  $H$  is a flat  $R$ -module, hence it has finite projective dimension (since  $R$  is  $m$ -perfect). Then,  $\text{Ext}(M, H) = 0$ . Thus, we have the following exact sequence:

$$0 \rightarrow \text{Hom}(M, H) \rightarrow \text{Hom}(M, P^0) \xrightarrow{\text{Hom}(M, f)} \text{Hom}(M, F) \rightarrow \text{Ext}(M, H) = 0.$$

Then, there exists  $\overline{\varphi} : M \rightarrow P^0$  such that  $\varphi = f\overline{\varphi}$ . Since  $\varphi$  is injective,  $\overline{\varphi}$  is also injective, and so we obtain the following short exact sequence of  $R$ -modules:

$$(*) \quad 0 \rightarrow M \xrightarrow{\overline{\varphi}} P^0 \rightarrow G^0 \rightarrow 0.$$

Now, to complete the proof, it remains to prove that  $\text{Ext}^i(G^0, F') = 0$  for all  $i > 0$  and all  $R$ -modules  $F'$  with finite projective dimension.

First, assume that  $F'$  is projective. Since  $\varphi$  is a flat preenvelope of  $M$ , there exists, for all  $\alpha \in \text{Hom}(M, F')$ , a homomorphism  $g : F \rightarrow F'$  such that  $\alpha = g\varphi$ , hence  $\alpha = gf\overline{\varphi}$ . This means that the functor  $\text{Hom}(-, F')$  leaves the short sequence  $(*)$  exact. Then, by the long exact sequence

$$0 \rightarrow \text{Hom}(G^0, F') \rightarrow \text{Hom}(P^0, F') \rightarrow \text{Hom}(M, F') \rightarrow \text{Ext}(G^0, F') \rightarrow \text{Ext}(P^0, F') = 0,$$

we deduce that  $\text{Ext}(G^0, F') = 0$ . Also, we use the short exact sequence  $(*)$  to deduce that  $\text{Ext}^i(G^0, F') = 0$  for all  $i > 0$  and all projective  $R$ -modules  $F'$ . Finally, this implies directly that  $\text{Ext}^i(G^0, F') = 0$  for all  $i > 0$  and all  $R$ -modules  $F'$  with finite projective dimension. ■

**Proof of Theorem 2.1.** First, from [12, Theorem 7.2.5 (2)] and [21, Theorem 2.28], the inequality  $\text{cot.D}(R) \leq \text{G-gldim}(R)$  holds for any arbitrary ring  $R$ .

Then, we prove the inequality  $\text{G-gldim}(R) \leq \text{G-wdim}(R) + \text{cot.D}(R)$  when  $R$  is coherent. For that, we may assume that  $\text{cot.D}(R) = m$  and  $\text{G-wdim}(R) = n$  are finite (i.e.,  $R$  is  $m$ -perfect and  $n$ -FC). Let  $M$  be an  $R$ -module, and consider an exact sequence of  $R$ -modules:

$$0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where each  $P_i$  is projective, and, from [4, Theorem 4.11],  $K_n$  is Gorenstein flat. We have:

$$(*) \quad \text{Ext}^{n+k}(M, Q) \cong \text{Ext}^k(K_n, Q) \quad \text{for all } k \geq 1 \text{ and all modules } Q.$$

Assume that  $Q$  is a projective  $R$ -module, and consider an exact sequence of  $R$ -modules:

$$0 \rightarrow Q \rightarrow C_0 \rightarrow \cdots \rightarrow C_{m-1} \rightarrow C_m \rightarrow 0,$$

where  $C_i$  is injective for  $i = 1, \dots, m-1$ , and then, from [12, Proposition 7.2.1],  $C_m$  is cotorsion. We have:

$$(**) \quad \text{Ext}^{m+i}(K_n, Q) \cong \text{Ext}^i(K_n, C_m) \quad \text{for all } i \geq 1.$$

Since  $\text{G-wdim}(R)$  is finite, each of the  $R$ -modules  $C_0, \dots, C_{m-1}$  has finite flat dimension (from [4, Theorem 4.11]). Then,  $C_m$  has finite flat dimension. Thus,  $\text{Ext}^i(K_n, C_m) = 0$  for all  $i \geq 1$  (from [21, Proposition 3.22] and since  $K_n$  is Gorenstein flat). Then, by  $(*)$  and  $(**)$ ,  $\text{Ext}^{n+m+i}(M, Q) = 0$  for all  $i \geq 1$ . This implies, from Lemma 2.2, that  $\text{Gpd}(M) \leq n + m$ , as desired. ■

Theorem 2.1 enables us to compute the Gorenstein global dimension of some particular cases of trivial extensions of rings and of group rings.

Recall that the trivial extension of a ring  $R$  by an  $R$ -module  $M$  is the ring denoted by  $R \ltimes M$  whose underlying group is  $A \times M$  with multiplication given by  $(r, m)(r', m') = (rr', rm' + r'm)$  (see for instance [19] and [20, Chapter 4, Section 4]). Next result compute the Gorenstein global dimension of a particular case of trivial extensions of rings. For that, we use the notion of finitistic projective dimension of rings. Recall the finitistic projective dimension of a ring  $R$ , denoted by  $\text{FPD}(R)$ , is defined by:

$$\text{FPD}(R) = \sup\{\text{pd}_R(M) \mid M \text{ } R\text{-module with } \text{pd}_R(M) < \infty\}.$$

From [21, Theorem 2.28], we have for every ring  $R$ :  $\text{FPD}(R) \leq \text{G-gldim}(R)$ , with equality if  $\text{G-gldim}(R)$  is finite.

**Proposition 2.5** *Let  $R \ltimes R$  be the trivial extension of a ring  $R$  by  $R$ . Then,  $\text{FPD}(R \ltimes R) = \text{FPD}(R)$ ,  $\text{cot.D}(R \ltimes R) = \text{cot.D}(R)$ , and  $\text{gldim}(R \ltimes R) = \infty$ .*

*Furthermore, if  $R$  is coherent, then  $\text{G-gldim}(R \ltimes R) = \text{G-gldim}(R)$ .*

The proof of this theorem involves the following results:

**Lemma 2.6** ([19], Theorem 4.28 and Remark page 81) *Let  $R$  be a ring and let  $M$  be any non-zero cyclic  $R$ -module. Then,  $\text{FPD}(R \ltimes M) = \sup\{\text{pd}_R(N) < \infty : \text{Tor}_i^R(M, N) = 0 \text{ for all } i > 0\}$  and  $\text{gldim}(R \ltimes M) = \infty$ .*

*In particular, if  $F$  is an  $R \ltimes M$ -module having finite projective dimension, then  $\text{pd}_{R \ltimes M}(F) = \text{pd}_R(R \otimes_{R \ltimes M} F)$ .*

**Lemma 2.7** ([19], Theorem 4.32) *Let  $R$  be a ring and let  $M$  be an  $R$ -module such that:*

$$\text{Ext}_R^i(M, M) \cong \begin{cases} R & \text{if } i=0; \\ 0 & \text{if } i>0. \end{cases}$$

*Then,  $\text{id}_{R \ltimes M}(R \ltimes M) = \text{id}_R(M)$ .*

**Proof of Proposition 2.5.** From Lemma 2.6,  $\text{FPD}(R \ltimes R) = \text{FPD}(R)$  and  $\text{gldim}(R \ltimes R) = \infty$ .

We prove that  $\text{cot.D}(R \ltimes R) = \text{cot.D}(R)$ . From Lemma 2.6, we have:

$$\text{pd}_{(R \ltimes R)}(F) = \text{pd}_R(R \otimes_{(R \ltimes R)} F)$$

for every  $R \ltimes R$ -module  $F$  with finite projective dimension. This implies that  $\text{cot.D}(R \ltimes R) \leq \text{cot.D}(R)$ . Conversely, consider a flat  $R$ -module  $F$ , then  $F \otimes_R (R \ltimes R)$  is a flat  $R \ltimes R$ -module. Thus, since  $R \ltimes R$  is a free  $R$ -module such that  $R \ltimes R \cong_R R^2$  we have:

$$\text{pd}_R(F \otimes_R R) = \text{pd}_R(F \otimes_R (R \ltimes R)) \leq \text{pd}_{(R \ltimes R)}(F \otimes_R (R \ltimes R)) \leq \text{cot.D}(R \ltimes R).$$

Therefore,  $\text{cot.D}(R) \leq \text{cot.D}(R \ltimes R)$ , as desired.

Now, we assume that  $R$  is coherent, and we prove the equality  $\text{G-gldim}(R \ltimes R) = \text{G-gldim}(R)$ .

First assume that  $\text{G-gldim}(R)$  is finite. Then, by the reason above and from [21, Theorem 2.28],  $\text{FPD}(R \ltimes R) = \text{FPD}(R) = \text{G-gldim}(R)$  is finite. So  $\text{cot.D}(R \ltimes R)$  is finite. On the other hand, from Lemma 2.7,  $\text{id}_{(R \ltimes R)}(R \ltimes R) = \text{id}_R(R)$  which is finite (by [4, Lemma 3.3] and since  $\text{G-gldim}(R)$  is finite). Then, by [4, Theorem 4.11],  $\text{G-wdim}(R) = \text{FP-id}_R(R) \leq \text{id}_R(R)$  is finite. Then, from Theorem 2.1,  $\text{G-gldim}(R \ltimes R)$  is finite. Therefore, from [21, Theorem 2.28],  $\text{G-gldim}(R \ltimes R) = \text{FPD}(R \ltimes R) = \text{G-gldim}(R)$ .

Similarly we show that  $\text{G-gldim}(R \ltimes R) = \text{G-gldim}(R)$  when  $\text{G-gldim}(R \ltimes R)$  is finite, and this gives the desired result. ■

As mentioned in the introduction, the Noetherian rings of finite Gorenstein global dimension are the same Iwanaga-Gorenstein rings; and in the class of rings of finite weak dimension the global dimension and the Gorenstein global dimension coincide. In the following example, we construct a family of non-Noetherian coherent rings  $\{S_i\}_{i \geq 1}$  such that  $\text{G-gldim}(S_i) = i$  and  $\text{wdim}(S_i) = \infty$  for every  $i \geq 1$ .

**Example 2.8** *Let  $R_n = R[X_1, X_2, \dots, X_n]$  be the polynomial ring in  $n$  indeterminates over a non-Noetherian hereditary ring  $R$ . Let  $S_i = R_{i-1} \ltimes R_{i-1}$  be the trivial extension of  $R_{i-1}$  by  $R_{i-1}$  for  $i \geq 1$  (such that  $R_0 = R$ ). Then, for every  $i \geq 1$ ,  $S_i$  is a non-Noetherian coherent ring with  $\text{G-gldim}(S_i) = i$  and  $\text{wdim}(S_i) = \infty$ .*

**Proof.** From [20, Theorem 7.3.1],  $R_n = R[X_1, X_2, \dots, X_n]$  is coherent for every  $n \geq 1$ . And by Hilbert's syzygy theorem,  $\text{gldim}(R_n) = \text{gldim}(R) + n = 1 + n$ . Therefore, Proposition 2.5 implies that  $\text{G-gldim}(S_i) = i$  for every  $i \geq 1$ .

Finally,  $\text{wdim}(S_i) = \infty$  for every  $i \geq 1$  follows from [4, Proposition 3.11] and since  $\text{gldim}(S_i) = \infty$  from Proposition 2.5. ■

We end this paper with a study of the global cotorsion dimension of group rings, and then the Gorenstein global dimension of a particular group ring is computed.

Let  $R$  be a ring and let  $G$  be an abelian group written multiplicatively. The free  $R$ -module on the elements of  $G$  with multiplication induced by  $G$  is a ring, called group ring of  $G$  over  $R$  and denoted by  $RG$  (see for instance [20, Chapter 8, Section 2]).

In [26], we have that  $RG$  is perfect if and only if  $R$  is perfect and  $G$  is finite. Here, we set the following extension.

**Theorem 2.9** *Let  $R$  be a ring and let  $G$  be an abelian group. We have:*

$$\text{cot.D}(R) \leq \text{cot.D}(RG) \leq \text{cot.D}(R) + \text{pd}_{RG}(R).$$

*Furthermore, if  $G$  and  $\text{pd}_{RG}(R)$  are finite, then  $\text{cot.D}(R) = \text{cot.D}(RG)$ .*

To prove this result we need the following lemma.

**Lemma 2.10** ([7], page 352) *Let  $R$  be a ring, let  $G$  be an abelian group, and let  $M$  and  $N$  be two  $RG$ -modules satisfying  $\text{Ext}_R^p(M, N) = 0$  for all  $p > 0$ . Then,*

$$\text{Ext}_{RG}^n(M, N) \cong \text{Ext}_{RG}^n(R, \text{Hom}_R(M, N))$$

*for all  $n > 0$ , where  $\text{Hom}_R(M, N)$  is the  $RG$ -module defined by  $(gf)(x) = g[f(g^{-1}x)]$  for  $x \in M$ ,  $f \in \text{Hom}_R(M, N)$ , and  $g \in G$ .*

**Proof of Theorem 2.9.** First, we prove the inequality  $\text{cot.D}(R) \leq \text{cot.D}(RG)$ . We may assume that  $\text{cot.D}(RG) = n$  is finite. Let  $F$  be a flat  $R$ -module, then  $F \otimes_R RG$  is a flat  $RG$ -module. Since  $RG \cong R^{(G)}$  is a free  $R$ -module,  $\text{pd}_R(F) = \text{pd}_R(F^{(G)}) = \text{pd}_R(F \otimes_R RG) \leq \text{pd}_{RG}(F \otimes_R RG) \leq n$ . This implies the desired inequality.

Now, we prove the inequality  $\text{cot.D}(RG) \leq \text{cot.D}(R) + \text{pd}_{RG}(R)$ . For that we may assume that  $\text{cot.D}(R) = s$  and  $\text{pd}_{RG}(R) = r$  are finite. Let  $F$  be a flat  $RG$ -module (then it is also flat as an  $R$ -module), and consider an exact sequence of  $RG$ -modules:

$$0 \rightarrow P_s \rightarrow \dots \rightarrow P_0 \rightarrow F \rightarrow 0,$$

where  $P_0, \dots, P_{s-1}$  are projective  $RG$ -modules, then they are projective as  $R$ -modules, and so  $P_s$  is a projective  $R$ -module (since  $\text{cot.D}(R) = s$ ). Thus,  $\text{Ext}_R^p(P_s, N) = 0$  for all  $p > 0$  and all  $R$ -modules  $N$ . Then, from Lemma 2.10 above and since  $\text{pd}_{RG}(R) = r$ ,

$$\text{Ext}_{RG}^n(P_s, N) \cong \text{Ext}_{RG}^n(R, \text{Hom}_R(P_s, N)) = 0$$

for all  $n > r$  and all  $RG$ -modules  $N$ . Thus,  $\text{pd}_{RG}(P_s) \leq r$  and so  $\text{pd}_{RG}(F) \leq s + r$ . Therefore,  $\text{cot.D}(RG) \leq s + r$ , as desired.

Assume now that  $G$  and  $\text{pd}_{RG}(R)$  are finite. From [6, Lemma 3.2 (a)],  $R$  is projective as an  $RG$ -module, and by the inequalities above,  $\text{cot.D}(R) = \text{cot.D}(RG)$ , as desired. ■

The above result and the main result (Theorem 2.1) are used to compute the Gorenstein global dimension of a particular group ring as follows:

**Proposition 2.11** *Let  $R$  be a ring with  $G\text{-wdim}(R) = 0$ . If  $G$  is a finite group such that its order is invertible in  $R$ , then  $G\text{-wdim}(RG) = 0$  and  $G\text{-gldim}(RG) = G\text{-gldim}(R)$ .*

**Proof.** First, note that  $R$  is coherent and so it is an IF-ring (from [8, Theorem 6] and [25, Proposition 4.2]). Then, from [11, Theorem 3 page 250],  $RG$  is an IF-ring and so  $G\text{-wdim}(RG) = 0$ .

Now, by Theorem 2.1,  $G\text{-gldim}(R) = \text{cot.D}(R)$  and  $G\text{-gldim}(RG) = \text{cot.D}(RG)$ . And from [20, Theorem 8.2.7],  $R$  is projective as  $RG$ -module. Thus, from Theorem 2.9,  $\text{cot.D}(RG) = \text{cot.D}(R)$ . This implies the desired equality  $G\text{-gldim}(RG) = G\text{-gldim}(R)$ . ■

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